**Property of Time-Invariance (Shift-Invariance) for a System Under Observation**

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When evaluating system properties, we treat a system as a closed box and analyze the relationships between input signals and their corresponding output signals. This process assumes that after inputting a signal, we can return the system to its original state.

A continuous-time system with input signal $x\left(t\right)$ and output signal $y\left(t\right)$ is time-invariant (shift-invariant) if whenever the input signal is delayed by $t\_{0}$ seconds, then the output signal will always be delayed by $t\_{0} $seconds as well for all real values of $t\_{0}.$

A way to visualize the time-invariance property is to show the equivalence between





That is, does $y\_{shifted}\left(t\right)=y\left(t-t\_{0}\right)$ for all possible real constant values of $t\_{0}$?

**One-sided infinite observation.** Let’s consider the system under observation for $t\geq 0^{-}$. Time $0^{-}$ means a time of 0 seconds before occurrence of a Dirac delta occurring at the origin. We can only observe $x\left(t\right)$ for $t\geq 0^{-}$ and $y\left(t\right)$ for $t\geq 0^{-}$. This means that we can only observe $x\left(t-t\_{0}\right)$ for $t\geq t\_{0}$and $y\left(t-t\_{0}\right)$ for $t\geq t\_{0}$.

**Example.** Consider a delay system that delays the input by *T* seconds, and we can only observe the input signal and the output signal for $t\geq 0^{-}$.

Conceptually, the delay block can be thought as a long wire that conducts electricity from the input to the output. Assuming electrons travel at 2/3 the speed of light, the length of the wire would be (2/3) *c* *T* where *c* is the speed of light (3 x 108 m/s). Such an implementation would be impractical, but nonetheless helpful in analyzing the system.

The first observed output value $y\left(0\right)$ would be due to the initial conditions in the delay system. In fact, the first *T* seconds of the output would due solely to the initial conditions in the system. For input $x\left(t\right)$ and output $y\left(t\right), $once the initial conditions have been output, $y\left(T\right)=x(0)$. That is, it takes *T* seconds for an input value (voltage) to arrive at the output.

The initial conditions for the delay system consist of the voltage values at different points in the wire at *t* = 0. Let’s denote these voltage values *v*(*t*) for –*T* < *t* ≤ 0. That is, $v\left(0\right)$ will be first, and $v\left(-T\right)$ will be the last, value among the initial conditions to be output. The spatial location for the voltage *v*(*t*) for –*T* < *t* ≤ 0 is (-2/3) *c* *t* meters from the output location.

Let *x*(*t*) = 0 for 0 ≤ *t* < *T* and 1 for *t* ≥ *T*. For input *x*(*t*), the output is

$$y\left(t\right)=\left[\begin{matrix}v\left(-t\right)&for 0 \leq t < T\\x\left(t–T\right)&for t \geq T\end{matrix}=\right.\left[\begin{matrix}v\left(-t\right)&for 0\leq t<T\\0&for T\leq t<2T\\1&for t\geq 2T\end{matrix}\right.$$

Let’s keep the same initial conditions, i.e. *v*(*t*) for –*T* < *t* ≤ 0, and the same definition for signal *x*(*t*). Now, we input $x(t-t\_{0})$ into the delay system

$$x\left(t-t\_{0}\right)=\left[\begin{matrix}unobserved&for 0 \leq t < t\_{0}\\0&for t\_{0} \leq t < T+t\_{0}\\1&for t \geq T+t\_{0}\end{matrix}\right.$$

and the output is

$$y\_{shifted}\left(t\right)=\left[\begin{matrix}\begin{matrix}v\left(-t\right)\\unobserved\end{matrix}&\begin{matrix}for 0 \leq t < T\\for T\leq t < T+t\_{0}\end{matrix}\\\begin{matrix}0\\1\end{matrix}&\begin{matrix}for T+t\_{0}\leq t<2T+t\_{0}\\for t \geq 2 T+t\_{0}\end{matrix}\end{matrix}\right.=\left[\begin{matrix}v\left(-t\right)&for 0 \leq t < T\\unobserved&for T\leq t < T+t\_{0}\\y\left(t-t\_{0}\right)&for t \geq T+t\_{0}\end{matrix}\right.$$

$y\_{shifted}\left(t\right)$ only equals $y\left(t-t\_{0}\right)$ for $t\geq T+t\_{0}$ because the initial conditions did not shift in time even though the input did.

Plots of the signals are given next followed by an analysis of initial conditions:

If the system were time-invariant, then $y\_{shift}\left(t\right)=y\left(t-t\_{0}\right)$ for all real $t\_{0}$ and $t\geq 0^{-}$. This holds for $for t \geq T+t\_{0}$. For $0 \leq t < T+t\_{0}$, all unobserved values and initial conditions would have to be equal to a constant value.